

Finite-dimensional irreducible modules of the universal Askey-Wilson algebra

Hau-wen Huang

Abstract

Let \mathbb{F} denote a field and fix a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. The universal Askey-Wilson algebra is an associative unital \mathbb{F} -algebra $\Delta = \Delta_q$ defined by generators and relations. The generators are A, B, C and the relations assert that each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

is central in Δ . We classify up to isomorphism the finite-dimensional irreducible Δ -modules, provided \mathbb{F} is algebraically closed and q is not a root of unity.

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1 Introduction

The Askey-Wilson algebra $AW(3)$ was introduced by Zhedanov in 1991 [9]. The algebra $AW(3)$ involves a nonzero scalar q and five parameters $\omega, \varrho, \varrho^*, \eta, \eta^*$. Given these data, the corresponding algebra $AW(3)$ is defined by generators K_0, K_1, K_2 and relations

$$\begin{aligned} qK_0K_1 - q^{-1}K_1K_0 &= K_2, \\ qK_1K_2 - q^{-1}K_2K_1 &= \omega K_1 + \varrho K_0 + \eta^*, \\ qK_2K_0 - q^{-1}K_0K_2 &= \omega K_0 + \varrho^* K_1 + \eta. \end{aligned}$$

In the article [9], Zhedanov displayed a family of finite-dimensional irreducible $AW(3)$ -modules. The structure of these modules can be described using the notion of a Leonard pair. Roughly speaking, a Leonard pair is a pair of diagonalizable linear transformations on a nonzero finite-dimensional vector space, each of which acts in an irreducible tridiagonal fashion on an eigenbasis for the other one [5]. Essentially, Zhedanov showed that on the above $AW(3)$ -modules the generators K_0, K_1 act as a Leonard pair. Terwilliger and Vidunas gave a more comprehensive description of how Leonard pairs are related to $AW(3)$ -modules. They showed that the underlying vector space of any Leonard pair A, B supports an irreducible $AW(3)$ -module on which K_0, K_1 act as A, B respectively [8, Theorem 1.5]. Conversely, let V denote a finite-dimensional irreducible $AW(3)$ -module on which each of K_0, K_1 is diagonalizable with all eigenspaces of dimension one. Then the pair K_0, K_1 acts on V as a Leonard pair, provided q is not a root of unity [8, Theorem 6.2].

The universal Askey-Wilson algebra $\Delta = \Delta_q$ was recently introduced by Terwilliger [6]. The algebra Δ is defined by three generators A, B, C and relations which assert that each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

is central in Δ . The algebra Δ is obtained from AW(3) by the following two-step procedure. First, the algebra is renormalized by a mild change of variables. Second, the remaining parameters are interpreted as central elements in the algebra (cf. [7]). By construction, every “most general” Askey-Wilson algebra AW(3) is a homomorphic image of Δ . In this paper, we classify up to isomorphism the finite-dimensional irreducible Δ -modules, provided the underlying field \mathbb{F} is algebraically closed and q is not a root of unity.

We now summarize our classification. It involves the quantum universal enveloping algebra $U = U_q(\mathfrak{sl}_2)$. The algebra U has a presentation (cf. [3]) with four generators $x, y^{\pm 1}, z$ subject to the following relations:

$$\begin{aligned} yy^{-1} &= 1, & y^{-1}y &= 1, \\ \frac{qxy - q^{-1}yx}{q - q^{-1}} &= 1, & \frac{qyz - q^{-1}zy}{q - q^{-1}} &= 1, & \frac{qzx - q^{-1}xz}{q - q^{-1}} &= 1. \end{aligned}$$

Given any nonzero scalars a, b, c in \mathbb{F} there exists a unique algebra homomorphism $\mathfrak{h}_{a,b,c} : \Delta \rightarrow U$ (cf. [7]) that sends

$$\begin{aligned} A &\mapsto xa + ya^{-1} + \frac{xy - yx}{q - q^{-1}}bc^{-1}, \\ B &\mapsto yb + zb^{-1} + \frac{yz - zy}{q - q^{-1}}ca^{-1}, \\ C &\mapsto zc + xc^{-1} + \frac{zx - xz}{q - q^{-1}}ab^{-1}. \end{aligned}$$

For an integer $d \geq 0$ let $V_{d,1}$ denote the $(d+1)$ -dimensional irreducible U -module of type 1. We pull back the U -module $V_{d,1}$ via $\mathfrak{h}_{a,b,c}$ and obtain a Δ -module which we denote by $V_d(a, b, c)$. Let M_d denote the set consisting of all triples $(a, b, c) \in \mathbb{F}^3$ that satisfy the following conditions:

(M1) $a \neq 0, b \neq 0, c \neq 0$.

(M2) None of $abc, a^{-1}bc, ab^{-1}c, abc^{-1}$ is among $q^{d-1}, q^{d-3}, \dots, q^{1-d}$.

We show that the Δ -module $V_d(a, b, c)$ is irreducible if and only if $(a, b, c) \in M_d$. Observe that if $(a, b, c) \in M_d$ then so is each of $(a^{-1}, b, c), (a, b^{-1}, c), (a, b, c^{-1})$. The three maps $M_d \rightarrow M_d$ given by

$$(a, b, c) \mapsto (a^{-1}, b, c), \quad (a, b, c) \mapsto (a, b^{-1}, c), \quad (a, b, c) \mapsto (a, b, c^{-1})$$

induce an action of the group $(\mathbb{Z}_2)^3$ on M_d . For $(a, b, c) \in M_d$ let $[a, b, c]$ denote the $(\mathbb{Z}_2)^3$ -orbit of M_d that contains (a, b, c) . We show that for (a, b, c) and (a', b', c') in M_d , if $[a, b, c] = [a', b', c']$ then the Δ -modules $V_d(a, b, c)$ and $V_d(a', b', c')$ are isomorphic. Let $(\mathbb{Z}_2)^3 \backslash M_d$ denote the set of $(\mathbb{Z}_2)^3$ -orbits of M_d . Let \mathcal{M}_d denote the set of the isomorphism classes of irreducible Δ -modules that have dimension $d+1$. In our main result (Theorem 4.6), we show that the map

$$\begin{aligned} (\mathbb{Z}_2)^3 \backslash M_d &\rightarrow \mathcal{M}_d \\ [a, b, c] &\mapsto \text{the isomorphism class of } V_d(a, b, c) \end{aligned}$$

is a bijection. The above result gives our classification of the finite-dimensional irreducible Δ -modules. In some related results, we determine which of these Δ -modules give Leonard pairs and Leonard triples.

2 Preliminaries

We fix a field \mathbb{F} and a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. An \mathbb{F} -algebra means an associative \mathbb{F} -algebra with unit.

The *universal Askey-Wilson algebra* $\Delta = \Delta_q$ is an \mathbb{F} -algebra with three generators A, B, C and the relations assert that each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} \quad (1)$$

is central in Δ (cf. [6, Definition 1.2]). We recall from [6] some facts concerning with the universal Askey-Wilson algebra Δ . For the central elements in (1), multiply each by $q + q^{-1}$ to get α, β, γ . Thus

$$\begin{aligned} A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} &= \frac{\alpha}{q + q^{-1}}, \\ B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} &= \frac{\beta}{q + q^{-1}}, \\ C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} &= \frac{\gamma}{q + q^{-1}}. \end{aligned}$$

The elements A, B, γ generate Δ . The elements C, α, β are expressed in terms of A, B, γ as follows:

$$C = \frac{\gamma}{q + q^{-1}} - \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}, \quad (2)$$

$$\alpha = \frac{B^2A - (q^2 + q^{-2})BAB + AB^2 + (q^2 - q^{-2})^2A + (q - q^{-1})^2B\gamma}{(q - q^{-1})(q^2 - q^{-2})}, \quad (3)$$

$$\beta = \frac{A^2B - (q^2 + q^{-2})ABA + BA^2 + (q^2 - q^{-2})^2B + (q - q^{-1})^2A\gamma}{(q - q^{-1})(q^2 - q^{-2})}.$$

Recall the notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{for all } n \geq 0.$$

A presentation of Δ is given by the generators A, B, γ and the following relations

$$\begin{aligned} A^3B - [3]A^2BA + [3]ABA^2 - BA^3 &= -(q^2 - q^{-2})^2(AB - BA), \\ B^3A - [3]B^2AB + [3]BAB^2 - AB^3 &= -(q^2 - q^{-2})^2(BA - AB), \\ A^2B^2 - B^2A^2 + (q^2 + q^{-2})(BABA - ABAB) &= -(q - q^{-1})^2(AB - BA)\gamma, \\ \gamma A &= A\gamma, \quad \gamma B = B\gamma. \end{aligned} \quad (4)$$

Lemma 2.1. [6, Theorem 4.1]. *The monomials $B^i C^j A^k \alpha^r \beta^s \gamma^t$ for all $i, j, k, r, s, t \geq 0$ form a basis of the \mathbb{F} -vector space Δ .*

To obtain Δ -modules one way is to pull back $U_q(\mathfrak{sl}_2)$ -modules via the known homomorphisms $\Delta \rightarrow U_q(\mathfrak{sl}_2)$ given in [7]. To describe these homomorphisms we recall some facts about $U_q(\mathfrak{sl}_2)$. The quantum algebra $U = U_q(\mathfrak{sl}_2)$ is an \mathbb{F} -algebra defined by generators $e, f, k^{\pm 1}$ and the following relations

$$kk^{-1} = k^{-1}k = 1, \quad ke = q^2ek, \quad kf = q^{-2}fk, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

The elements $e, f, k^{\pm 1}$ are called the *Chevalley generators* of U and the above presentation of U is called the *Chevalley presentation* (cf. [4, Section 1.1]). An central element of U is called the *Casimir element* defined by

$$\Phi = ef + \frac{q^{-1}k + qk^{-1}}{(q - q^{-1})^2}.$$

A *normalized Casimir element* [7] is given by

$$\begin{aligned} \Lambda &= (q - q^{-1})^2 \Phi \\ &= (q - q^{-1})^2 ef + q^{-1}k + qk^{-1}. \end{aligned} \tag{5}$$

By [3, Theorem 2.1] the \mathbb{F} -algebra U is isomorphic to the \mathbb{F} -algebra defined by generators $x, y^{\pm 1}, z$ and the following relations

$$\begin{aligned} yy^{-1} &= 1, & y^{-1}y &= 1, \\ \frac{qxy - q^{-1}yx}{q - q^{-1}} &= 1, & \frac{qyz - q^{-1}zy}{q - q^{-1}} &= 1, & \frac{qzx - q^{-1}xz}{q - q^{-1}} &= 1. \end{aligned}$$

An isomorphism with the Chevalley presentation of U is given by

$$y^{\pm 1} \mapsto k^{\pm 1}, \quad z \mapsto k^{-1} + f(q - q^{-1}), \quad x \mapsto k^{-1} - ek^{-1}q^{-1}(q - q^{-1}).$$

The inverse of this isomorphism is given by

$$k^{\pm 1} \mapsto y^{\pm 1}, \quad f \mapsto (z - y^{-1})(q - q^{-1})^{-1}, \quad e \mapsto (1 - xy)q(q - q^{-1})^{-1}.$$

The elements $x, y^{\pm 1}, z$ are called the *equitable generators* of U and the above presentation for U is called the *equitable presentation* (cf. [3, Definition 2.2]).

By [7, Proposition 1.1], for any nonzero scalars $a, b, c \in \mathbb{F}$ there exists a unique \mathbb{F} -algebra homomorphism $\mathfrak{h}_{a,b,c} : \Delta \rightarrow U$ that sends

$$A \mapsto xa + ya^{-1} + \frac{xy - yx}{q - q^{-1}}bc^{-1}, \tag{6}$$

$$B \mapsto yb + zb^{-1} + \frac{yz - zy}{q - q^{-1}}ca^{-1}, \tag{7}$$

$$C \mapsto zc + xc^{-1} + \frac{zx - xz}{q - q^{-1}}ab^{-1}. \tag{8}$$

Moreover the homomorphism $\mathfrak{h}_{a,b,c}$ sends

$$\alpha \mapsto \Lambda(a + a^{-1}) + (b + b^{-1})(c + c^{-1}), \tag{9}$$

$$\beta \mapsto \Lambda(b + b^{-1}) + (c + c^{-1})(a + a^{-1}), \tag{10}$$

$$\gamma \mapsto \Lambda(c + c^{-1}) + (a + a^{-1})(b + b^{-1}). \tag{11}$$

3 A finite dimensional Δ -module $V_d(a, b, c)$

For the rest of this paper we fix an integer $d \geq 0$. Moreover assume that \mathbb{F} is algebraically closed and q is not a root of unity.

Lemma 3.1. [4, Theorem 2.6]. *For each $\varepsilon \in \{1, -1\}$ there exists an irreducible U -module $V_{d,\varepsilon}$ with basis $\{\mathbf{u}_i\}_{i=0}^d$ such that*

$$\begin{aligned} k\mathbf{u}_i &= \varepsilon q^{d-2i} \mathbf{u}_i & (0 \leq i \leq d), \\ f\mathbf{u}_i &= [i+1] \mathbf{u}_{i+1} & (0 \leq i \leq d-1), \quad f\mathbf{u}_d = 0, \\ e\mathbf{u}_i &= \varepsilon [d-i+1] \mathbf{u}_{i-1} & (1 \leq i \leq d), \quad e\mathbf{u}_0 = 0. \end{aligned}$$

Every irreducible U -module with dimension $d+1$ is isomorphic to $V_{d,1}$ or $V_{d,-1}$.

Referring to Lemma 3.1 the parameter ε is called the type of $V_{d,\varepsilon}$. If \mathbb{F} has characteristic two then $V_{d,1}$ and $V_{d,-1}$ are isomorphic. By (5) the normalized Casimir element Λ of U acts on $V_{d,\varepsilon}$ as the scalar

$$\varepsilon(q^{d+1} + q^{-d-1}). \quad (12)$$

By [3, Lemma 4.2] there is a basis $\{u_i\}_{i=0}^d$ of $V_{d,\varepsilon}$ on which x, y, z act as follows.

$$\varepsilon y u_i = q^{d-2i} u_i \quad (0 \leq i \leq d), \quad (13)$$

$$(\varepsilon z - q^{2i-d}) u_i = (q^{-d} - q^{2i-d+2}) u_{i+1} \quad (0 \leq i \leq d-1), \quad (\varepsilon z - q^d) u_d = 0, \quad (14)$$

$$(\varepsilon x - q^{2i-d}) u_i = (q^d - q^{2i-d-2}) u_{i-1} \quad (1 \leq i \leq d), \quad (\varepsilon x - q^{-d}) u_0 = 0. \quad (15)$$

By [3, Theorem 7.5] there exists an invertible linear transformation Ω on $V_{d,\varepsilon}$ such that

$$\Omega^{-1} x \Omega = y, \quad \Omega^{-1} y \Omega = z, \quad \Omega^{-1} z \Omega = x.$$

Let $\{v_i\}_{i=0}^d$ denote the basis of $V_{d,\varepsilon}$ to which Ω sends $\{u_i\}_{i=0}^d$. The elements x, y, z act on $\{v_i\}_{i=0}^d$ as follows.

$$\varepsilon x v_i = q^{d-2i} v_i \quad (0 \leq i \leq d), \quad (16)$$

$$(\varepsilon y - q^{2i-d}) v_i = (q^{-d} - q^{2i-d+2}) v_{i+1} \quad (0 \leq i \leq d-1), \quad (\varepsilon y - q^d) v_d = 0, \quad (17)$$

$$(\varepsilon z - q^{2i-d}) v_i = (q^d - q^{2i-d-2}) v_{i-1} \quad (1 \leq i \leq d), \quad (\varepsilon z - q^{-d}) v_0 = 0. \quad (18)$$

Let $\{w_i\}_{i=0}^d$ denote the basis of $V_{d,\varepsilon}$ to which Ω sends $\{v_i\}_{i=0}^d$. The elements x, y, z act on $\{w_i\}_{i=0}^d$ as follows.

$$\varepsilon z w_i = q^{d-2i} w_i \quad (0 \leq i \leq d), \quad (19)$$

$$(\varepsilon x - q^{2i-d}) w_i = (q^{-d} - q^{2i-d+2}) w_{i+1} \quad (0 \leq i \leq d-1), \quad (\varepsilon x - q^d) w_d = 0, \quad (20)$$

$$(\varepsilon y - q^{2i-d}) w_i = (q^d - q^{2i-d-2}) w_{i-1} \quad (1 \leq i \leq d), \quad (\varepsilon y - q^{-d}) w_0 = 0. \quad (21)$$

Until further notice we assume that a, b, c are arbitrary nonzero scalars taken from \mathbb{F} . The U -module $V_{d,1}$ supports a Δ -module given by

$$Xu = X^{\natural_{a,b,c}} u \quad (22)$$

for all $X \in \Delta$ and all $u \in V_{d,1}$. We denote the Δ -module by $V_d(a, b, c)$. Note that in the above, if we replace $V_{d,1}$ by $V_{d,-1}$ then the resulting Δ -module is isomorphic to $V_d(-a, -b, -c)$.

By (9)–(12) the central elements α, β, γ of Δ act on $V_d(a, b, c)$ as the following scalars

$$\xi_\alpha = (q^{d+1} + q^{-d-1})(a + a^{-1}) + (b + b^{-1})(c + c^{-1}), \quad (23)$$

$$\xi_\beta = (q^{d+1} + q^{-d-1})(b + b^{-1}) + (c + c^{-1})(a + a^{-1}), \quad (24)$$

$$\xi_\gamma = (q^{d+1} + q^{-d-1})(c + c^{-1}) + (a + a^{-1})(b + b^{-1}), \quad (25)$$

respectively. We define

$$\theta_i = aq^{2i-d} + a^{-1}q^{d-2i} \quad (0 \leq i \leq d), \quad (26)$$

$$\theta_i^* = bq^{2i-d} + b^{-1}q^{d-2i} \quad (0 \leq i \leq d), \quad (27)$$

$$\theta_i^\varepsilon = cq^{2i-d} + c^{-1}q^{d-2i} \quad (0 \leq i \leq d). \quad (28)$$

Let t_1, t_2, t_3 denote mutually commuting indeterminates. Define

$$\sigma_i(t_1, t_2, t_3) = (q^{d-i+1} - q^{i-d-1})(t_1q^{i-1} - t_2t_3^{-1}q^{d-i}) \quad (1 \leq i \leq d),$$

$$\tau_i(t_1, t_2, t_3) = (q^i - q^{-i})(t_1^{-1}t_3q^{1-i} - t_2^{-1}q^{i-d}) \quad (1 \leq i \leq d).$$

By (6), (7) and (13)–(15) the action of A, B on the basis $\{u_i\}_{i=0}^d$ of $V_d(a, b, c)$ is as follows.

$$(A - \theta_i)u_i = \sigma_i(a, b, c)u_{i-1} \quad (1 \leq i \leq d), \quad (A - \theta_0)u_0 = 0, \quad (29)$$

$$(B - \theta_{d-i}^*)u_i = \tau_{i+1}(a, b, c)u_{i+1} \quad (0 \leq i \leq d-1), \quad (B - \theta_0^*)u_d = 0. \quad (30)$$

By (6), (8) and (16)–(18) the action of A, C on the basis $\{v_i\}_{i=0}^d$ of $V_d(a, b, c)$ is as follows.

$$(C - \theta_i^\varepsilon)v_i = \sigma_i(c, a, b)v_{i-1} \quad (1 \leq i \leq d), \quad (C - \theta_0^\varepsilon)v_0 = 0, \quad (31)$$

$$(A - \theta_{d-i})v_i = \tau_{i+1}(c, a, b)v_{i+1} \quad (0 \leq i \leq d-1), \quad (A - \theta_0)v_d = 0. \quad (32)$$

By (7), (8) and (19)–(21) the action of B, C on the basis $\{w_i\}_{i=0}^d$ of $V_d(a, b, c)$ is as follows.

$$(B - \theta_i^*)w_i = \sigma_i(b, c, a)w_{i-1} \quad (1 \leq i \leq d), \quad (B - \theta_0^*)w_0 = 0, \quad (33)$$

$$(C - \theta_{d-i}^\varepsilon)w_i = \tau_{i+1}(b, c, a)w_{i+1} \quad (0 \leq i \leq d-1), \quad (C - \theta_0^\varepsilon)w_d = 0. \quad (34)$$

By (29)–(31) the characteristic polynomials of A, B, C on $V_d(a, b, c)$ are

$$K_A(t) = \prod_{i=0}^d (t - \theta_i), \quad (35)$$

$$K_B(t) = \prod_{i=0}^d (t - \theta_i^*), \quad (36)$$

$$K_C(t) = \prod_{i=0}^d (t - \theta_i^\varepsilon), \quad (37)$$

respectively. By (35)–(37) the following lemma is straightforward.

Lemma 3.2. *Let a, b, c denote any nonzero scalars taken from \mathbb{F} . Then the traces of A, B, C on $V_d(a, b, c)$ are*

$$\text{tr}A = (a + a^{-1})[d + 1], \quad (38)$$

$$\text{tr}B = (b + b^{-1})[d + 1], \quad (39)$$

$$\text{tr}C = (c + c^{-1})[d + 1], \quad (40)$$

respectively. Moreover a, b, c are the roots of the following quadratic polynomials

$$[d + 1]t^2 - \text{tr}At + [d + 1], \quad (41)$$

$$[d + 1]t^2 - \text{tr}Bt + [d + 1], \quad (42)$$

$$[d + 1]t^2 - \text{tr}Ct + [d + 1], \quad (43)$$

respectively.

For $0 \leq i \leq d$ we define U_i, V_i, W_i to be the one-dimensional subspaces of $V_d(a, b, c)$ spanned by u_i, v_i, w_i respectively. The following example shows that Δ -modules are incompletely reducible in general. Let $d = 1$ and consider the Δ -module $V_d(a, b, c)$ associated with $a = q, b = q^2, c = q$. The subspace U_1 is a submodule of $V_d(a, b, c)$ since $\sigma_1(a, b, c) = 0$. The subspace U_0 is the unique A -invariant complement of U_1 since $\theta_0 \neq \theta_1$. However U_0 is not B -invariant since $\tau_1(a, b, c) \neq 0$. Therefore $V_d(a, b, c)$ is incompletely reducible.

The goal of the rest of this section is to give necessary and sufficient conditions on a, b, c for the Δ -module $V_d(a, b, c)$ to be irreducible. We begin with an obvious necessary condition. For each $1 \leq i \leq d$ define $\phi_i(t_1, t_2, t_3) = \sigma_i(t_1, t_2, t_3)\tau_i(t_1, t_2, t_3)$ and its explicit form is

$$\phi_i(t_1, t_2, t_3) = t_1^{-1}t_2q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})(q^{-i} - t_1t_2^{-1}t_3q^{i-d-1})(q^{-i} - t_1t_2^{-1}t_3^{-1}q^{i-d-1}).$$

Observe that

$$\phi_i(t_1, t_2, t_3) = \phi_i(t_1, t_2, t_3^{-1}) \quad (1 \leq i \leq d). \quad (44)$$

Lemma 3.3. *If $V_d(a, b, c)$ is an irreducible Δ -module then the following condition holds:*

(N1) *The scalars $\phi_i(a, b, c), \phi_i(c, a, b), \phi_i(b, c, a)$ for all $1 \leq i \leq d$ are nonzero.*

Proof. If there exists $1 \leq i \leq d$ such that $\sigma_i(a, b, c) = 0$ (resp. $\tau_i(a, b, c) = 0$) then by (29) and (30),

$$\sum_{h=i}^d U_h \quad \left(\text{resp.} \quad \sum_{h=0}^{i-1} U_h \right)$$

is a submodule of $V_d(a, b, c)$, a contradiction. Therefore $\phi_i(a, b, c) \neq 0$ for all $1 \leq i \leq d$. By similar arguments $\phi_i(c, a, b) \neq 0$ and $\phi_i(b, c, a) \neq 0$ for all $1 \leq i \leq d$. \square

Lemma 3.4. *The following statements hold:*

- (i) *If $\phi_i(a, b, c) \neq 0$ for all $1 \leq i \leq d$ then the Δ -module $V_d(a, b, c)$ is isomorphic to $V_d(a, b, c^{-1})$.*
- (ii) *If $\phi_i(c, a, b) \neq 0$ for all $1 \leq i \leq d$ then the Δ -module $V_d(a, b, c)$ is isomorphic to $V_d(a, b^{-1}, c)$.*

(iii) If $\phi_i(b, c, a) \neq 0$ for all $1 \leq i \leq d$ then the Δ -module $V_d(a, b, c)$ is isomorphic to $V_d(a^{-1}, b, c)$.

Proof. We show (i). For each $1 \leq i \leq d$ the scalars $\tau_i(a, b, c)$, $\tau_i(a, b, c^{-1})$ are nonzero by (44). Let $\epsilon_0 = 1$ and

$$\epsilon_i = \prod_{h=1}^i \frac{\tau_h(a, b, c)}{\tau_h(a, b, c^{-1})} \quad (1 \leq i \leq d).$$

Let $\mathbf{u}_i = \epsilon_i u_i$ for all $0 \leq i \leq d$. By (29), (30) the action of A, B on the basis $\{\mathbf{u}_i\}_{i=0}^d$ of $V_d(a, b, c)$ is as follows.

$$\begin{aligned} (A - \theta_i)\mathbf{u}_i &= \sigma_i(a, b, c^{-1})\mathbf{u}_{i-1} & (1 \leq i \leq d), & & (A - \theta_0)\mathbf{u}_0 &= 0, \\ (B - \theta_{d-i}^*)\mathbf{u}_i &= \tau_{i+1}(a, b, c^{-1})\mathbf{u}_{i+1} & (0 \leq i \leq d-1), & & (B - \theta_0^*)\mathbf{u}_d &= 0. \end{aligned}$$

On $V_d(a, b, c^{-1})$ the central element $\gamma \in \Delta$ acts as the scalar ξ_γ since ξ_γ is unchanged if we replace c by c^{-1} . Therefore $V_d(a, b, c)$ and $V_d(a, b, c^{-1})$ are isomorphic since A, B, γ generate Δ . By similar arguments (ii) and (iii) follow. \square

Lemma 3.5. *If $V_d(a, b, c)$ is an irreducible Δ -module then the following conditions hold:*

(N2) *The scalars $\phi_i(a, b, c^{-1})$, $\phi_i(c^{-1}, a, b)$, $\phi_i(b, c^{-1}, a)$ for all $1 \leq i \leq d$ are nonzero.*

(N3) *The scalars $\phi_i(a, b^{-1}, c)$, $\phi_i(c, a, b^{-1})$, $\phi_i(b^{-1}, c, a)$ for all $1 \leq i \leq d$ are nonzero.*

(N4) *The scalars $\phi_i(a^{-1}, b, c)$, $\phi_i(c, a^{-1}, b)$, $\phi_i(b, c, a^{-1})$ for all $1 \leq i \leq d$ are nonzero.*

Proof. By Lemma 3.3 we have (N1). Therefore $V_d(a, b, c)$ is isomorphic to each of $V_d(a, b, c^{-1})$, $V_d(a, b^{-1}, c)$, $V_d(a^{-1}, b, c)$ by Lemma 3.4. Applying Lemma 3.3 to $V_d(a, b, c^{-1})$, $V_d(a, b^{-1}, c)$ and $V_d(a^{-1}, b, c)$ we have (N2)–(N4), respectively. \square

Let M_d denote the set of all triples $(a, b, c) \in \mathbb{F}^3$ that satisfy the following conditions:

(M1) $a \neq 0, b \neq 0, c \neq 0$.

(M2) None of $abc, a^{-1}bc, ab^{-1}c, abc^{-1}$ is among $q^{d-1}, q^{d-3}, \dots, q^{1-d}$.

Lemma 3.6. *The Δ -module $V_d(a, b, c)$ is irreducible if and only if $(a, b, c) \in M_d$.*

Proof. (Necessity): By the construction of $V_d(a, b, c)$ condition (M1) holds. By Lemma 3.3 and Lemma 3.5 we have (N1)–(N4). These conditions are equivalent to (M2) since $q^{2i} \neq 1$ for all $1 \leq i \leq d$.

(Sufficiency): By (30) we have

$$(B - \theta_1^*)(B - \theta_2^*) \cdots (B - \theta_d^*)V_d(a, b, c) \subseteq U_d. \quad (45)$$

For each $0 \leq i, j \leq d$ let m_{ij} denote the scalar in \mathbb{F} such that

$$(B - \theta_1^*) \cdots (B - \theta_d^*)(A - \theta_0) \cdots (A - \theta_{i-1})u_j = m_{ij}u_d. \quad (46)$$

Here $(A - \theta_0) \cdots (A - \theta_{i-1})$ is interpreted as 1 if $i = 0$. By (29),

$$m_{ij} = 0 \quad (0 \leq j < i \leq d). \quad (47)$$

We claim that for each $0 \leq i \leq j \leq d$

$$m_{ij} = \prod_{h=1}^{j-i} (\theta_0^* - \theta_{d-h+1}^*) \prod_{h=1}^{d-j} \tau_{j+h}(a, b, c) \prod_{h=1}^i \frac{[d+h-j][j-h+1]}{[d+h-i][i-h+1]} \phi_h(a, b^{-1}, c). \quad (48)$$

The first, second, and third products in the right-hand side of (48) are interpreted to be 1 if $i = j$, $j = d$, and $i = 0$, respectively. Proceed by induction on i . Using (30), (46) it is routine to verify that (48) holds for $i = 0$. Now fix i, j with $1 \leq i \leq j \leq d$. Using (29), (46) we deduce that

$$m_{ij} = (\theta_j - \theta_{i-1})m_{i-1,j} + \sigma_j(a, b, c)m_{i-1,j-1}. \quad (49)$$

Evaluating the right-hand side of (49) by using induction hypothesis the claim follows.

Let V denote any nonzero Δ -submodule of $V_d(a, b, c)$. We show that $V = V_d(a, b, c)$. Pick any nonzero vector $u \in V$. Let $a_0, a_1, \dots, a_d \in \mathbb{F}$ such that $u = a_0 u_0 + a_1 u_1 + \cdots + a_d u_d$. By (45) there exists $b_0, b_1, \dots, b_d \in \mathbb{F}$ such that

$$(B - \theta_1^*) \cdots (B - \theta_d^*)(A - \theta_0) \cdots (A - \theta_{i-1})u = b_i u_d \quad (0 \leq i \leq d). \quad (50)$$

Since V is a Δ -module, for each $0 \leq i \leq d$ we have $b_i u_d \in V$. Expressing (50) in terms of matrices we obtain

$$\begin{pmatrix} m_{00} & m_{01} & \cdot & \cdot & \cdot & m_{0d} \\ m_{10} & m_{11} & \cdot & \cdot & \cdot & m_{1d} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ m_{d0} & m_{d1} & \cdot & \cdot & \cdot & m_{dd} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_d \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_d \end{pmatrix}.$$

The above $(d+1)$ by $(d+1)$ matrix, denoted by M , is upper triangular by (47). By (M2) and since $q^{2i} \neq 1$ for all $0 \leq i \leq d$ each diagonal entry of M is nonzero. Therefore M is invertible. Since the a_i ($0 \leq i \leq d$) are not all zero there exists $1 \leq i \leq d$ such that $b_i \neq 0$. This shows that U_d is a subspace of V . The space U_d generates $V_d(a, b, c)$ as a Δ -module since $\sigma_i(a, b, c) \neq 0$ for all $1 \leq i \leq d$ and by (29). Therefore $V = V_d(a, b, c)$. The result follows. \square

4 Classification of finite-dimensional irreducible Δ -modules

Observe that if $(a, b, c) \in M_d$ then so is each of

$$(a^{-1}, b, c), \quad (a, b^{-1}, c), \quad (a, b, c^{-1}).$$

The three maps $M_d \rightarrow M_d$ given by

$$(a, b, c) \mapsto (a^{-1}, b, c), \quad (a, b, c) \mapsto (a, b^{-1}, c), \quad (a, b, c) \mapsto (a, b, c^{-1})$$

induce an action of the group $(\mathbb{Z}_2)^3$ on M_d . For $(a, b, c) \in M_d$ let $[a, b, c]$ denote the $(\mathbb{Z}_2)^3$ -orbit of M_d that contains (a, b, c) . As a consequence of Lemma 3.4 we have

Corollary 4.1. *For (a, b, c) and (a', b', c') in M_d , if $[a, b, c] = [a', b', c']$ then the Δ -modules $V_d(a, b, c)$ and $V_d(a', b', c')$ are isomorphic.*

Let $(\mathbb{Z}_2)^3 \backslash M_d$ denote the set of all $(\mathbb{Z}_2)^3$ -orbits on M_d . Let \mathcal{M}_d denote the set of the isomorphism classes of irreducible Δ -modules that have dimension $d + 1$. By Corollary 4.1 we may define the map $\pi : (\mathbb{Z}_2)^3 \backslash M_d \rightarrow \mathcal{M}_d$ given by

$$[a, b, c] \mapsto \text{the isomorphism class of } V_d(a, b, c)$$

for all $[a, b, c] \in (\mathbb{Z}_2)^3 \backslash M_d$. Before proving that π is a bijection we need some auxiliary results.

Lemma 4.2. *Let λ denote a nonzero scalar in \mathbb{F} . Let*

$$\vartheta_i = \lambda q^{2i} + \lambda^{-1} q^{-2i} \quad \text{for all integers } i. \quad (51)$$

Then exactly one of the following conditions holds:

- (i) *The scalars ϑ_i for all integers i are mutually distinct.*
- (ii) *There exists an integer j such that $\vartheta_{\frac{j+i}{2}}$ for all positive integers $i \equiv j \pmod{2}$ are mutually distinct. Moreover $\vartheta_{\frac{j+i}{2}} = \vartheta_{\frac{j-i}{2}}$ for all integers $i \equiv j \pmod{2}$.*

Proof. Suppose that (i) does not hold. Then there exist two distinct integers k, ℓ such that $\vartheta_k = \vartheta_\ell$. Since $q^{2k-2\ell} \neq 1$ it follows that $\lambda^2 = q^{2k+2\ell}$. By this and since q is not a root of unity the integer $j = k + \ell$ satisfies (ii). \square

Define a left ideal $\mathcal{I}_d(a, b, c)$ of Δ generated by

$$A - \theta_0, \quad (52)$$

$$K_B(B), \quad (53)$$

$$(A - \theta_1)(B - \theta_d^*) - \phi_1(a, b, c), \quad (54)$$

$$\alpha - \xi_\alpha, \quad \beta - \xi_\beta, \quad \gamma - \xi_\gamma. \quad (55)$$

Lemma 4.3. *There exists a unique Δ -module homomorphism $\psi : \Delta/\mathcal{I}_d(a, b, c) \rightarrow V_d(a, b, c)$ that sends $1 + \mathcal{I}_d(a, b, c)$ to u_0 .*

Proof. By (23)–(25) the central elements α, β, γ of Δ act on $V_d(a, b, c)$ as $\xi_\alpha, \xi_\beta, \xi_\gamma$ respectively. Since $K_B(t)$ is the characteristic polynomial of B on $V_d(a, b, c)$ it follows that $K_B(B)$ vanishes on $V_d(a, b, c)$. By (29) and (30),

$$(A - \theta_0)u_0 = 0, \quad (A - \theta_1)(B - \theta_d^*)u_0 = \phi_1(a, b, c)u_0.$$

By the above comments the existence of ψ follows. The desired map is unique since $1 + \mathcal{I}_d(a, b, c)$ generates $\Delta/\mathcal{I}_d(a, b, c)$ as a Δ -module. \square

Lemma 4.4. *The \mathbb{F} -vector space $\Delta/\mathcal{I}_d(a, b, c)$ is spanned by $B^i + \mathcal{I}_d(a, b, c)$ for all $0 \leq i \leq d$.*

Proof. By Lemma 2.1 the monomials $B^i C^j A^k \alpha^r \beta^s \gamma^t + \mathcal{I}_d(a, b, c)$ for all $i, j, k, r, s, t \geq 0$ span $\Delta/\mathcal{I}_d(a, b, c)$. By (52) and (55) the monomial $B^i C^j A^k \alpha^r \beta^s \gamma^t + \mathcal{I}_d(a, b, c) \in \mathbb{F} B^i C^j + \mathcal{I}_d(a, b, c)$. We claim that for each $i, j \geq 0$

$$B^i C^j + \mathcal{I}_d(a, b, c) \in \sum_{k \geq 0} \mathbb{F} B^k + \mathcal{I}_d(a, b, c). \quad (56)$$

Proceed by induction on j . Clearly our claim holds for $j = 0$. Now fix $j \geq 1$. Using (2) we have

$$B^i C^j \in \mathbb{F} B^i C^{j-1} B A + \mathbb{F} B^i C^{j-1} A B + \mathbb{F} B^i C^{j-1} \gamma. \quad (57)$$

By (52),

$$B^i C^{j-1} B A + \mathcal{I}_d(a, b, c) \in \mathbb{F} B^i C^{j-1} B + \mathcal{I}_d(a, b, c). \quad (58)$$

By (52) and (54),

$$B^i C^{j-1} A B + \mathcal{I}_d(a, b, c) \in \mathbb{F} B^i C^{j-1} + \mathbb{F} B^i C^{j-1} B + \mathcal{I}_d(a, b, c). \quad (59)$$

By (55),

$$B^i C^{j-1} \gamma + \mathcal{I}_d(a, b, c) \in \mathbb{F} B^i C^{j-1} + \mathcal{I}_d(a, b, c). \quad (60)$$

Combining (57)–(60),

$$B^i C^j + \mathcal{I}_d(a, b, c) \in \mathbb{F} B^i C^{j-1} + \mathbb{F} B^i C^{j-1} B + \mathcal{I}_d(a, b, c). \quad (61)$$

Applying induction hypothesis to the right-hand side of (61) the claim follows. By (53) the right-hand side of (56) is equal to the subspace of $\Delta/\mathcal{I}_d(a, b, c)$ spanned by $B^k + \mathcal{I}_d(a, b, c)$ for all $0 \leq k \leq d$. The result follows. \square

As a consequence of Lemma 4.4 we have

Corollary 4.5. *The dimension of the \mathbb{F} -vector space $\Delta/\mathcal{I}_d(a, b, c)$ is less than or equal to $d + 1$.*

Our classification of the finite-dimensional irreducible Δ -modules is given below.

Theorem 4.6. *The map $\pi : (\mathbb{Z}_2)^3 \setminus M_d \rightarrow \mathcal{M}_d$ is a bijection.*

Proof. (Injection): Suppose that (a, b, c) and (a', b', c') are in M_d such that the Δ -modules $V_d(a, b, c)$ and $V_d(a', b', c')$ are isomorphic. Since A has the same trace on $V_d(a, b, c)$ and $V_d(a', b', c')$ denoted by $\text{tr} A$, by Lemma 3.2 each of a, a' is a root of (41). Therefore $a' = a^{\pm 1}$. Similarly $b' = b^{\pm 1}$ and $c' = c^{\pm 1}$. This shows that $[a, b, c] = [a', b', c']$ and therefore π is injective.

(Surjection): Let V denote an irreducible Δ -module with dimension $d + 1$. We show that there exist nonzero scalars $a, b, c \in \mathbb{F}$ such that $V_d(a, b, c)$ is isomorphic to V . If so, then $(a, b, c) \in M_d$ by Lemma 3.6 and we are done. For any $\theta \in \mathbb{F}$ and any $X \in \Delta$ we let

$$V_X(\theta) = \{v \in V \mid Xv = \theta v\}.$$

Since \mathbb{F} is algebraically closed and the dimension of V is finite, there exists $\vartheta \in \mathbb{F}$ such that $V_A(\vartheta) \neq 0$. Let λ denote a nonzero scalar in \mathbb{F} such that $\vartheta = \lambda + \lambda^{-1}$. Consider the infinite sequence (51). By Lemma 4.2 there are only finitely many integers i satisfying $V_A(\vartheta_i) \neq 0$. Therefore there exists a nonzero scalar $\mu = \lambda q^{2i}$ for some integer i such that

$$V_A(\mu q^{-2} + \mu^{-1} q^2) = 0, \quad V_A(\mu + \mu^{-1}) \neq 0. \quad (62)$$

Similarly there exists a nonzero scalar $\nu \in \mathbb{F}$ such that

$$V_B(\nu q^{-2} + \nu^{-1} q^2) = 0, \quad V_B(\nu + \nu^{-1}) \neq 0.$$

We choose $a = \mu q^d$ and $b = \nu q^d$. Since \mathbb{F} is algebraically closed and V is irreducible each central element of Δ acts on V as a scalar. Since $q^{2d+2} \neq -1$ we may choose a nonzero $c \in \mathbb{F}$ for which γ acts on V as the scalar ξ_γ shown in (25). To see that V is isomorphic to $V_d(a, b, c)$, we shall show that there exists a Δ -module isomorphism $\chi : \Delta/\mathcal{I}_d(a, b, c) \rightarrow V$. If so, we obtain a Δ -module homomorphism $\psi \circ \chi^{-1} : V \rightarrow V_d(a, b, c)$, where ψ is from Lemma 4.3. By the irreducibility of V the kernel of $\psi \circ \chi^{-1}$ is trivial and hence an isomorphism, as required.

Let $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ be as in (26) and (27), respectively. Applying any $u \in V_A(\theta_0)$ to either side of (4), we factor the resulting equation and by (62) we obtain that

$$(A - \theta_0)(A - \theta_1)Bu = 0.$$

Therefore $V_A(\theta_0)$ is $(A - \theta_1)B$ -invariant. Since \mathbb{F} is algebraically closed there exists an eigenvector $\mathbf{u}_0 \in V_A(\theta_0)$ of the linear map $u \mapsto (A - \theta_1)Bu$ for $u \in V_A(\theta_0)$. Similarly there exists an eigenvector $\mathbf{u}_0^* \in V_B(\theta_0^*)$ of the linear map $u \mapsto (B - \theta_1^*)Au$ for $u \in V_B(\theta_0^*)$. Let

$$\mathbf{u}_i = (B - \theta_{d-i+1}^*)\mathbf{u}_{i-1} \quad (1 \leq i \leq d), \quad (63)$$

$$\mathbf{u}_i^* = (A - \theta_{d-i+1})\mathbf{u}_{i-1}^* \quad (1 \leq i \leq d). \quad (64)$$

For $0 \leq i \leq d$ let U_i and U_i^* denote the subspaces of V spanned by \mathbf{u}_i and \mathbf{u}_i^* , respectively. We now show that

$$(A - \theta_i)\mathbf{u}_i \in \sum_{h=0}^{i-1} U_h \quad (0 \leq i \leq d). \quad (65)$$

For $i = 0$ the right hand side of (65) is interpreted as 0. Proceed by induction on i . By the choice of \mathbf{u}_0 we have (65) for $i = 0, 1$. Observe that

$$\theta_{i-1} - (q^2 + q^{-2})\theta_i + \theta_{i+1} = 0 \quad (1 \leq i \leq d-1). \quad (66)$$

Suppose $i \geq 2$. Apply \mathbf{u}_{i-2} to either side of (3). Using (63), (66) and induction hypothesis to simplify the resulting equation, we obtain (65). Similarly

$$(B - \theta_i^*)\mathbf{u}_i^* \in \sum_{h=0}^{i-1} U_h^* \quad (0 \leq i \leq d). \quad (67)$$

By (65), for each $0 \leq i \leq d$ the space $\sum_{h=0}^i U_h$ is A -invariant. If there exists $0 \leq i \leq d-1$ such that $\mathbf{u}_{i+1} \in \sum_{h=0}^i U_h$ then $\sum_{h=0}^i U_h$ is B -invariant by (63), a contradiction to the irreducibility of

V . Therefore $\{\mathbf{u}_i\}_{i=0}^d$ is a basis of V . Similarly $\{\mathbf{u}_i^*\}_{i=0}^d$ is a basis of V . Now, by (65) and (67) the characteristic polynomials of A and B on V are $K_A(t)$ and $K_B(t)$ as shown in (35) and (36), respectively. In particular

$$K_B(B)\mathbf{u}_0 = 0. \quad (68)$$

Let $b_0, b_1, \dots, b_d \in \mathbb{F}$ such that $B\mathbf{u}_d = b_0\mathbf{u}_0 + b_1\mathbf{u}_1 + \dots + b_d\mathbf{u}_d$. Let \mathbf{B} denote the matrix representing B with respect to $\{\mathbf{u}_i\}_{i=0}^d$. Let \mathbf{I} denote the matrix representing the identity map on V with respect to $\{\mathbf{u}_i\}_{i=0}^d$. Then

$$\det(t\mathbf{I} - \mathbf{B}) = K_B(t). \quad (69)$$

Eliminating the common factors of either side of (69) and substituting $t = \theta_d^*, \theta_{d-1}^*, \dots, \theta_0^*$ into the resulting equation, we successively find that $b_i = 0$ for $0 \leq i \leq d-1$ and $b_d = \theta_0^*$. Therefore

$$(B - \theta_0^*)\mathbf{u}_d = 0. \quad (70)$$

Let $\{\phi_i\}_{i=1}^d$ denote the sequence of scalars in \mathbb{F} such that

$$(A - \theta_i)\mathbf{u}_i \in \phi_i\mathbf{u}_{i-1} + \sum_{h=0}^{i-2} U_h \quad (1 \leq i \leq d). \quad (71)$$

Let $1 \leq i \leq d$ be given. Apply \mathbf{u}_{i-1} to either side of (3) and use (63), (70), (71) to simplify the resulting equation. Comparing the coefficient of \mathbf{u}_i on either side we obtain that

$$\begin{aligned} \phi_{i-1} - (q^2 + q^{-2})\phi_i + \phi_{i+1} &= (q^2 + q^{-2})(\theta_i\theta_{d-i}^* + \theta_{i-1}\theta_{d-i+1}^*) \\ &\quad - (\theta_{i-1} + \theta_i)(\theta_{d-i}^* + \theta_{d-i+1}^*) - (q - q^{-1})^2\xi_\gamma \end{aligned} \quad (1 \leq i \leq d),$$

where we interpret $\phi_0 = 0$ and $\phi_{d+1} = 0$. The scalars $\phi_i = \phi_i(a, b, c)$ for all $1 \leq i \leq d$ satisfy the above recurrence relations. Using $q^4 \neq 1$ and $q^{4d+4} \neq 1$ we deduce that no other scalars ϕ_i ($1 \leq i \leq d$) satisfy these recurrence relations. In particular

$$(A - \theta_1)(B - \theta_d^*)\mathbf{u}_0 = \phi_1(a, b, c)\mathbf{u}_0. \quad (72)$$

Similarly

$$(B - \theta_1^*)(A - \theta_d)\mathbf{u}_0^* = \phi_1(b, a, c)\mathbf{u}_0^*. \quad (73)$$

Now apply \mathbf{u}_0^* to either side of (3) and use (67) with $i = 0$ and (73) to simplify the resulting equation. Comparing the coefficient of \mathbf{u}_0^* on either side we deduce that α acts on V as the scalar ξ_α shown in (23). By a similar argument the central element β of Δ acts on V as the scalar ξ_β shown in (24). In particular

$$\alpha\mathbf{u}_0 = \xi_\alpha\mathbf{u}_0, \quad \beta\mathbf{u}_0 = \xi_\beta\mathbf{u}_0, \quad \gamma\mathbf{u}_0 = \xi_\gamma\mathbf{u}_0. \quad (74)$$

By (65) with $i = 0$, (68), (72) and (74) there exists a unique Δ -module homomorphism $\chi : \Delta/\mathcal{I}_d(a, b, c) \rightarrow V$ that sends $1 + \mathcal{I}_d(a, b, c)$ to \mathbf{u}_0 . By the irreducibility of V the homomorphism χ is surjective. Comparing the dimension of $\Delta/\mathcal{I}_d(a, b, c)$ and V using Corollary 4.5, we have χ is an isomorphism. The result follows. \square

Let $d \geq 0$ be an integer. Any $(d + 1)$ -dimensional irreducible Δ -module V has the following properties:

- (a) Let $\text{tr}A$, $\text{tr}B$, $\text{tr}C$ denote the traces of A , B , C on V respectively. By Lemma 3.2 and Theorem 4.6 the Δ -module V is isomorphic to $V_d(a, b, c)$ if and only if a , b , c are the roots of (41)–(43) respectively. For the rest of this paper fix $a, b, c \in \mathbb{F}$ such that V is isomorphic to $V_d(a, b, c)$. Moreover $(a, b, c) \in M_d$.
- (b) Let $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$, $\{\theta_i^\varepsilon\}_{i=0}^d$ be as in (26)–(28) respectively. The characteristic polynomials of A , B , C on V are (35)–(37) respectively. Let $0 \leq i \leq d$ be given. By (29) and since $(a, b, c) \in M_d$ the linear transformation $A - \theta_i$ on V has rank d . Therefore the eigenspace of A on V associated with θ_i is one-dimensional. Similarly the eigenspace of B (resp. C) on V associated with θ_i^* (resp. θ_i^ε) is one-dimensional.
- (c) As a consequence of (b) the following are equivalent:
 - (i) A (resp. B) (resp. C) is diagonalizable on V .
 - (ii) $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) (resp. $\{\theta_i^\varepsilon\}_{i=0}^d$) are mutually distinct.
 - (iii) a^2 (resp. b^2) (resp. c^2) is not among $q^{2d-2}, q^{2d-4}, \dots, q^{2-2d}$.
- (d) A square matrix is said to be *tridiagonal* if each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. A tridiagonal matrix is said to be *irreducible* if each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. A pair of linear transformations on V is called a *Leonard pair* [5] if for each of the two transformations, there exists a basis of V with respect to which the matrix representing that transformation is diagonal and the matrix representing the other transformation is irreducible tridiagonal. By (b), (c) and [8, Theorem 6.2] the following are equivalent:
 - (i) A, B (resp. A, C) (resp. B, C) act on V as a Leonard pair.
 - (ii) Each of A, B (resp. A, C) (resp. B, C) is diagonalizable on V .
 - (iii) Neither of a^2, b^2 (resp. a^2, c^2) (resp. b^2, c^2) is among $q^{2d-2}, q^{2d-4}, \dots, q^{2-2d}$.
- (e) A triple of linear transformations on V is called a *Leonard triple* [1] if for each of these transformations, there exists a basis of V with respect to which the matrix representing that transformation is diagonal and the matrices representing the other two transformations are irreducible tridiagonal. By (c), (d) and [2, Theorem 14.5] the following are equivalent:
 - (i) A, B, C act on V as a Leonard triple.
 - (ii) Any two of A, B, C act on V as a Leonard pair.
 - (iii) Each of A, B, C is diagonalizable on V .
 - (iv) None of a^2, b^2, c^2 is among $q^{2d-2}, q^{2d-4}, \dots, q^{2-2d}$.
- (f) A sequence $\{U_i\}_{i=0}^d$ of one-dimensional subspaces of V is called a *decomposition* of V if

$$V = \sum_{i=0}^d U_i \quad (\text{direct sum}).$$

(i) There exists a unique decomposition $\{U_i\}_{i=0}^d$ of V such that

$$(A - \theta_i)U_i = U_{i-1} \quad (1 \leq i \leq d), \quad (A - \theta_0)U_0 = 0, \quad (75)$$

$$(B - \theta_{d-i}^*)U_i = U_{i+1} \quad (0 \leq i \leq d-1), \quad (B - \theta_0^*)U_d = 0. \quad (76)$$

(ii) There exists a unique decomposition $\{V_i\}_{i=0}^d$ of V such that

$$(C - \theta_i^\varepsilon)V_i = V_{i-1} \quad (1 \leq i \leq d), \quad (C - \theta_0^\varepsilon)V_0 = 0,$$

$$(A - \theta_{d-i})V_i = V_{i+1} \quad (0 \leq i \leq d-1), \quad (A - \theta_0)V_d = 0.$$

(iii) There exists a unique decomposition $\{W_i\}_{i=0}^d$ of V such that

$$(B - \theta_i^*)W_i = W_{i-1} \quad (1 \leq i \leq d), \quad (B - \theta_0^*)W_0 = 0,$$

$$(C - \theta_{d-i}^\varepsilon)W_i = W_{i+1} \quad (0 \leq i \leq d-1), \quad (C - \theta_0^\varepsilon)W_d = 0.$$

Proof. We show (i). The existence is immediate from (29) and (30). By (75) the space U_0 is a one-dimensional subspace of the eigenspace of A on V associated with θ_0 . By (b) the choice of U_0 is unique. Now, by (76) the choice of these spaces U_i for all $1 \leq i \leq d$ is unique. The uniqueness follows. By similar arguments (ii) and (iii) follow. \square

(g) There exist at most two U -module structures on V that satisfy (22) for all $X \in \Delta$ and $u \in V$. Any such U -module structure on V is irreducible since the Δ -module structure on V is irreducible. More precisely, there is exactly one such U -module structure on V which is isomorphic to $V_{d,1}$. Assume that the characteristic of \mathbb{F} is not two and the U -module structure on V is isomorphic to $V_{d,-1}$. If $a^2 = b^2 = c^2 = -1$ there is exactly one such U -module structure on V . If one of a^2, b^2, c^2 is not equal to -1 there is no such a U -module structure on V .

Proof. We first assume that the U -module structure on V is isomorphic to $V_{d,1}$. The existence is immediate from the construction of $V_d(a, b, c)$. By (13)–(21) and (29)–(34), on any such U -module V the eigenspaces of the equitable generators x, y, z of U associated with eigenvalue q^{d-2i} ($0 \leq i \leq d$) are the subspaces V_i, U_i, W_i of V from (f), respectively. The uniqueness follows. We now assume that the characteristic of \mathbb{F} is not two and the U -module structure on V is isomorphic to $V_{d,-1}$. By the comment below (22), as a Δ -module V is also isomorphic to $V_d(-a, -b, -c)$. If $a^2 = b^2 = c^2 = -1$, a similar argument to above shows that there exists a unique such U -module on V . If one of a^2, b^2, c^2 is not equal to -1 , there is no such a U -module structure on V since $V_d(a, b, c)$ and $V_d(-a, -b, -c)$ are not isomorphic by (a). The result follows. \square

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Hau-wen Huang
Mathematics Division
National Center for Theoretical Sciences
National Tsing-Hua University
Hsinchu 30013, Taiwan, R.O.C.
Email: hauwenh@math.cts.nthu.edu.tw